CHAPTER 13 FREQUENCY RESPONSE ANALYSIS (NYQUIST PLOT)

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After completing this chapter, the students will be able to:

- Sketch a Nyquist diagram,
- Use the Nyquist criterion to determine the stability of open-loop and closed-loop systems
- Find stability and gain and phase margins using Nyquist diagrams.

1. Introduction

The performance of a control system is more realistically measured by its time-domain characteristics. The reason is that the performance of most control systems is judged based on the time responses due to certain test signals. This is in contrast to the analysis and design of communication systems for which the frequency response is of more importance, since most of the signals to be processed are either sinusoidal or composed of sinusoidal components.

The time response of a control system is usually more difficult to determine analytically, especially for high-order systems. In design problems, there are no unified methods of arriving at a designed system that meets the time-domain performance specifications, such as maximum overshoot, rise time, delay time, settling time, and so on. On the other hand, in the frequency domain, there is a wealth of graphical methods available that are not limited to low-order systems. It is important to realize that there are correlating relations between the frequency-domain and the time-domain performances in a linear system, so the time-domain properties of the system can be predicted based on the frequency-domain characteristics. The frequency domain is also more convenient for measurements of system sensitivity to noise and parameter variations. With these concepts in mind, we consider the primary motivation for conducting control systems analysis and design in the frequency domain to be convenience and the availability of the existing analytical tools. Another reason is that it presents an alternative point of view to control-system problems, which often provides valuable or crucial information in the complex analysis and design of control systems.

The polar plot is the locus of vectors $|G(j\omega)| \angle G(j\omega)$ as ω is varied from zero to infinity. Note that in polar plots a positive (negative) phase angle is measured counter clockwise from the positive real axis. The polar plot is often called the Nyquist plot. An example of such a plot is shown in Fig. 1. Each point on the polar plot of G(jω) represents the terminal point of a vector at a particular value of ω. In the polar plot, it is important to show the frequency graduation of the locus. The projections of G(jω) on the real and imaginary axes are its real and imaginary components.

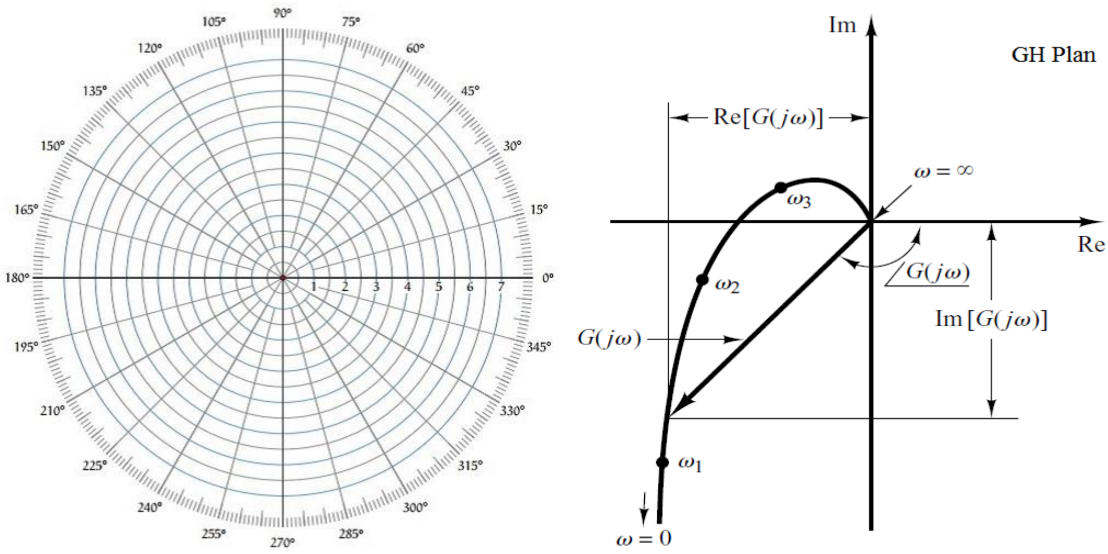


Fig. 1 Polar plot

Polar plot of Integral term:

The magnitude and angle of the integral term is represented as:

$$G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega} = \frac{1}{\omega}\angle -90^\circ$$

The polar plot is shown in Fig. 2. That is coincide with the negative imaginary axis.

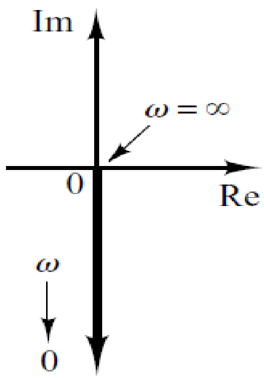


Fig. 2, Polar plot of integral term

Polar plot of Derivative term:

The magnitude and angle of the derivative term is represented as:

$$G(j\omega) = j\omega \quad \rightarrow \quad G(j\omega) = \omega\angle 90$$

The polar plot is coincide with the positive imaginary axis as shown in Fig. 3.

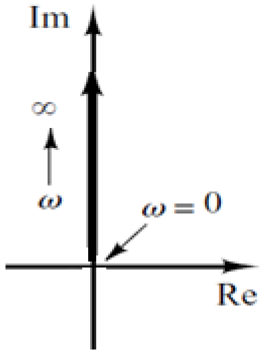


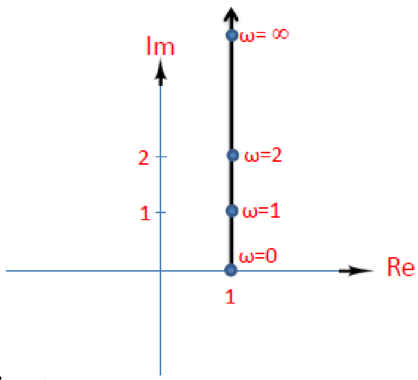
Fig. 3, Polar plot of derivative term

Polar plot of First-Order term:

A) Assuming that the first-order term is in numerator:

$$G(s) = S + 1 \quad \rightarrow G(j\omega) = 1 + j\omega$$

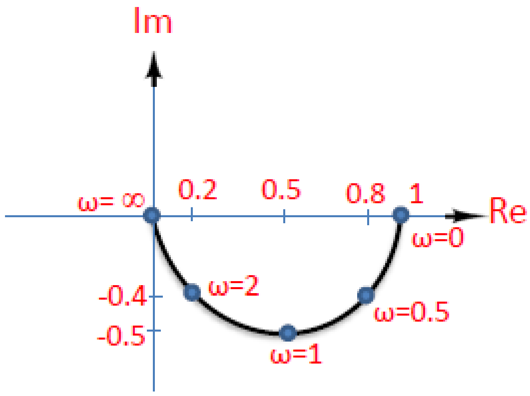
ω	Real part	Imag. part
0	1	0
1	1	1
2	1	2
3	1	3
4	1	4
…		
∞	1	∞



Assuming that the first-order term is in denominator:

$$G(s) = \frac{1}{S+1} \quad \rightarrow \quad G(j\omega) = \frac{1}{1+j\omega} \times \frac{1-j\omega}{1-j\omega} = \frac{1}{1+\omega^2} - j\frac{\omega}{1+\omega^2}$$

ω	Real part	Imag. part
0	1	0
0.5	0.8	−0.4
1	0.5	−0.5
2	0.2	−0.4
3	0.1	−0.3
…		
∞	0	0



In general,

$$G(s) = \frac{1}{TS+1} \quad \rightarrow \quad G(j\omega) = \frac{1}{1+j\omega T} \times \frac{1-j\omega T}{1-j\omega T} = \frac{1}{1+T^2\omega^2} - j\frac{\omega T}{1+T^2\omega^2}$$

$$G(j\omega) = \frac{1}{1 + j\omega T} = \frac{1}{\sqrt{1+\omega^2 T^2}} \angle -\tan^{-1}\omega T$$

The polar plot of this transfer function is a semicircle as the frequency ω is varied from zero to infinity, as shown in Fig. 3. The center is located at 0.5 on the real axis, and the radius is equal to 0.5.

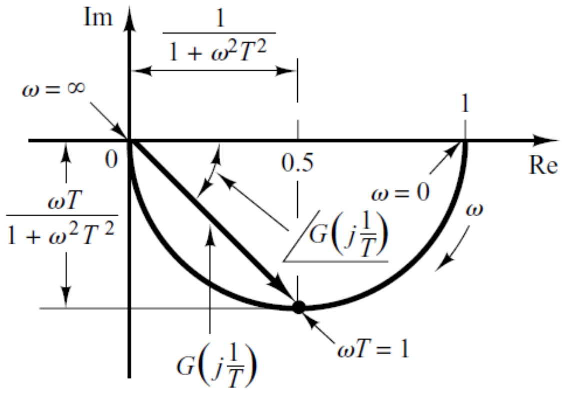


Fig. 3 Polar plot of first-order system

Polar plot of Second-Order term:

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}, \qquad \text{for } \zeta > 0$$

$$\lim_{\omega \to 0} G(j\omega) = 1\underline{/0^\circ} \quad \text{and} \quad \lim_{\omega \to \infty} G(j\omega) = 0\underline{/-180^\circ}$$

The polar plot of underdamped, second-order transfer function starts at 1∠0 and ends at 0∠-180 as ω increases from zero to infinity. Thus, the high-frequency portion of G(jω) is tangent to the negative real axis as shown in Fig. 4.

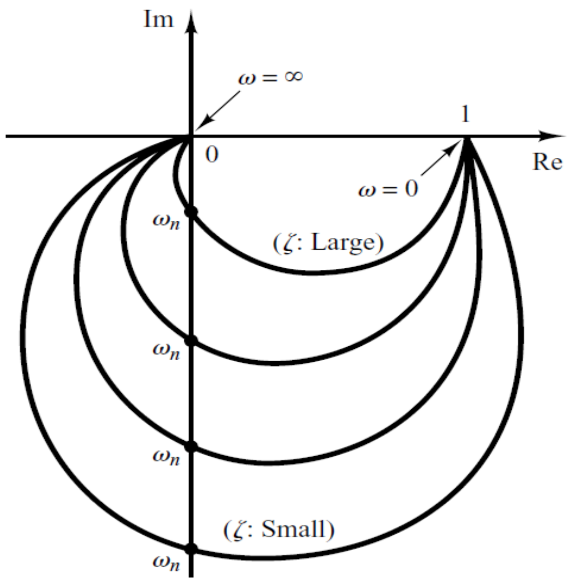


Fig. 4 Polar plot of second-order term

For the underdamped case at $\omega=\omega_n$, we have $G(j\omega_n)=1/(j2\zeta)$, and the phase angle at $\omega = \omega_n$ is -90°. Therefore, it can be seen that the frequency at which the $G(j\omega)$ locus intersects the imaginary axis is the undamped natural frequency ω_n. In the polar plot, the frequency point whose distance from the origin is maximum corresponds to the resonant frequency ω_r. The peak value of $G(j\omega)$ is obtained as the ratio of the magnitude of the vector at the resonant frequency ω_r to the magnitude of the vector at $\omega = 0$.

Polar plot is defined as the locus of the magnitude of the loop T.F. $|G(j\omega)H(j\omega)|$ and the angle $\angle G(j\omega)H(j\omega)$ in the GH plan as ω varies from 0 to ∞. We can obtain $G(j\omega)H(j\omega)$ from $G(s)H(s)$ by replacing s by jω

Nyquist plot is defined as the locus of the magnitude of the loop T.F. $|G(j\omega)H(j\omega)|$ and the angle $\angle G(j\omega)H(j\omega)$ in the GH plan as ω varies from -∞ to ∞.

When draw polar / Nyquist plot, we must consider the following points:

1- Plot is free hand,
2- Compute $|G(j\omega)H(j\omega)|$ and $\angle G(j\omega)H(j\omega)$ at ω = 0 and ∞ only,
3- Calculate the point(s) of intersection of the plot with real and imaginary axis of GH plan and the corresponding value of ω. Taking in consideration type (0) systems don't intersect with real or imaginary axes

Example (1):

Draw the polar plot for the system whose open loop T.F. is

$$G(s)H(s) = \frac{10}{S+1}$$

Step #1: Replace each S by jω, then find an expression for the magnitude and phase

$$G(j\omega)H(j\omega) = \frac{10}{1+j\omega}$$

$$|G(j\omega)H(j\omega)| = \frac{10}{\sqrt{1+\omega^2}}; \ \angle G(j\omega)H(j\omega) = -tan^{-1}(\frac{\omega}{1})$$

Step #2: Calculate the magnitude and angle of GH at ω = 0

$$|G(j\omega)H(j\omega)| = 10; \ \angle G(j\omega)H(j\omega) = 0$$

Step #3: Calculate the magnitude and angle of GH at ω → ∞

$$|G(j\omega)H(j\omega)| = 0; \ \angle G(j\omega)H(j\omega) = -90$$

Step #4: Intersection with real axis

No intersection as the system is type (0)

Step #5: intersection with imaginary axix

No intersection as the system is type (0)

The polar plot is as shown in Fig. 5.

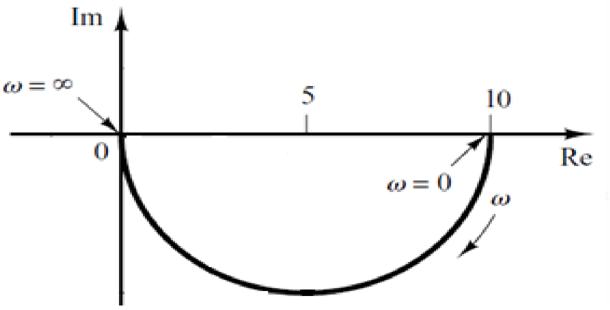


Fig. 5, Polar plot of example (1)

Example (2):

Draw the polar plot for the system whose open loop T.F. is

$$G(s)H(s) = \frac{10(S+5)}{S+1}$$

Step #1: Replace each S by jω, then find an expression for the magnitude and phase

$$G(j\omega)H(j\omega) = \frac{10(5+j\omega)}{1+j\omega}$$

$$|G(j\omega)H(j\omega)| = \frac{10\sqrt{25+\omega^2}}{\sqrt{1+\omega^2}}; \ \angle G(j\omega)H(j\omega) = tan^{-1}(\frac{\omega}{5}) - tan^{-1}(\frac{\omega}{1})$$

Step #2: Calculate the magnitude and angle of GH at ω = 0

$$|G(j\omega)H(j\omega)| = 50; \ \angle G(j\omega)H(j\omega) = 0$$

Step #3: Calculate the magnitude and angle of GH at ω → ∞

$$|G(j\omega)H(j\omega)| = 10; \ \angle G(j\omega)H(j\omega) = 0$$

Step #4: Intersection with real axis

No intersection as the system is type (0)

Step #5: intersection with imaginary axix

No intersection as the system is type (0)

The polar plot is as shown in Fig. 6

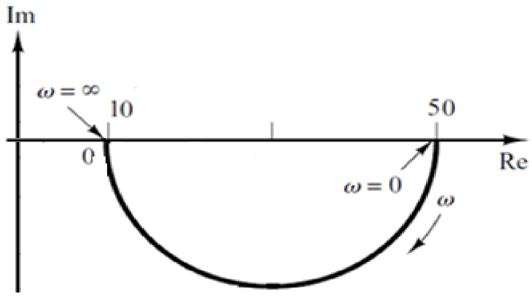


Fig. 6, Polar plot of example (2)

For *type* 0 *systems:* The starting point of the polar plot (which corresponds to ω=0) is finite and is on the positive real axis. The tangent to the polar plot at ω=0 is perpendicular to the real axis. The terminal point, which corresponds to ω=∞, is on real axis, and the curve is tangent to one of the axes according to the number of poles.

Example (3):

Draw the polar plot for the system whose open loop T.F. is

$$G(s)H(s) = \frac{10}{S(S+1)}$$

Step #1: Replace each S by jω, then find an expression for the magnitude and phase

$$G(j\omega)H(j\omega) = \frac{10}{j\omega(1+j\omega)}$$

$$|G(j\omega)H(j\omega)| = \frac{10}{\omega\sqrt{1+\omega^2}};\ \angle G(j\omega)H(j\omega) = -90 - tan^{-1}(\frac{\omega}{1})$$

$$G(j\omega)H(j\omega) = \frac{10}{j\omega(1+j\omega)} = \frac{10}{-\omega^2 + j\omega} \times \frac{-\omega^2 - j\omega}{-\omega^2 - j\omega} = \frac{-10}{\omega^2+1} - j\frac{10}{\omega^3+\omega}$$

Step #2: Calculate the magnitude and angle of GH at ω = 0

$$|G(j\omega)H(j\omega)| = \infty;\ \angle G(j\omega)H(j\omega) = -90$$

Step #3: Calculate the magnitude and angle of GH at ω → ∞

$$|G(j\omega)H(j\omega)| = 0;\ \angle G(j\omega)H(j\omega) = -180$$

Step #4: Intersection with real axis

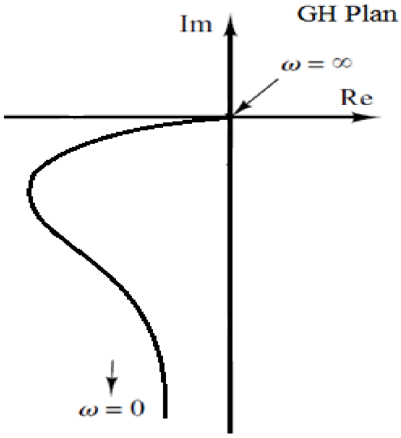
As the system is type (1): to get the intersection with real, take the imaginary part = 0

$$\frac{10}{\omega^3+\omega} = 0 \ \rightarrow \ \omega = \infty \quad (no\ intersection)$$

Step #5: intersection with imaginary axix

As the system is type (1): to get the intersection with imaginary, take the real part = 0

$$\frac{-10}{\omega^2+1} = 0 \rightarrow \omega = \infty \quad (no\ intersection)$$



Example (4):

Draw the polar plot for the system whose open loop T.F. is

$$G(s)H(s) = \frac{1}{S(S+1)(S+0.5)}$$

Step #1: Replace each S by jω, then find an expression for the magnitude and phase

$$G(j\omega)H(j\omega) = \frac{1}{j\omega(1+j\omega)(0.5+j\omega)}$$

$$|G(j\omega)H(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}\sqrt{0.25+\omega^2}}; \ \angle G(j\omega)H(j\omega) = -90 - tan^{-1}(\frac{\omega}{1}) - tan^{-1}(\frac{\omega}{0.5})$$

$$G(j\omega)H(j\omega) = \frac{1}{j\omega(1+j\omega)(0.5+j\omega)}$$

$$= \frac{1}{-1.5\omega^2 + j(-\omega^3+0.5\omega)} \times \frac{-1.5\omega^2 - j(-\omega^3+0.5\omega)}{-1.5\omega^2 - j(-\omega^3+0.5\omega)}$$

$$= \frac{-1.5\omega^2}{2.25\omega^4 + (0.5\omega-\omega^3)^2} - j\frac{-\omega^3+0.5\omega}{2.25\omega^4+(0.5\omega-\omega^3)^2}$$

Step #2: Calculate the magnitude and angle of GH at ω = 0

$$|G(j\omega)H(j\omega)| = \infty; \ \angle G(j\omega)H(j\omega) = -90$$

Step #3: Calculate the magnitude and angle of GH at ω → ∞

$$|G(j\omega)H(j\omega)| = 0; \ \angle G(j\omega)H(j\omega) = -270$$

Step #4: Intersection with real axis

As the system is type (1): to get the intersection with real, take the imaginary part = 0

$$\frac{-\omega^3 + 0.5\omega}{2.25\omega^4 + (0.5\omega - \omega^3)^2} = 0 \rightarrow \omega^2 = 0.5 \ \rightarrow \ \omega_c = 0.707$$

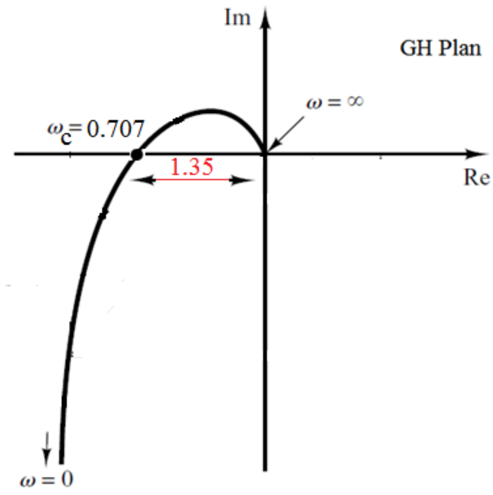
Real distance can be obtained by substituting the value of ω_c in the real part

$$real\ distance = \frac{-1.5 \times 0.5}{2.25 \times 0.25} = -1.35$$

Step #5: intersection with imaginary axix

As the system is type (1): to get the intersection with imaginary, take the real part = 0

$$\frac{-1.5\omega^2}{2.25\omega^4 + (0.5\omega - \omega^3)^2} = 0 \ \rightarrow \ \ \omega = \infty \quad (no\ intersection)$$



For *type* 1 *systems:* the jω term in the denominator contributes ‑90° to the total phase angle of G(jω). At ω=0, the magnitude of G(jω) is infinity, and the phase angle becomes ‑90°. At low frequencies, the polar plot is asymptotic to a line parallel to the negative imaginary axis. At ω=∞, the magnitude becomes zero, and the curve converges to the origin and is tangent to one of the axes.

2. Nyquist Criterion

Because the Nyquist criterion is a graphical method, we need to establish the concepts of encircled and enclosed, which are used for the interpretation of the Nyquist plots for stability.

Encircled:

A point is said to be encircled by a closed path if it is found inside the path.

For example, point A in Fig. 5(a) is encircled by the closed path Γ, because A is *inside* the closed path. Point *B* is not encircled by the closed path Γ, because it is located *outside* the path.

Enclosed:

A point is said to be enclosed by a closed path if it lies to the left of the path when the path is traversed in CCW direction.

The concept of enclosure is particularly useful if only a portion of the closed path is shown. For example, point A in Fig. 5(b) is *enclosed* by Γ. However, point *B* isn't *enclosed.*

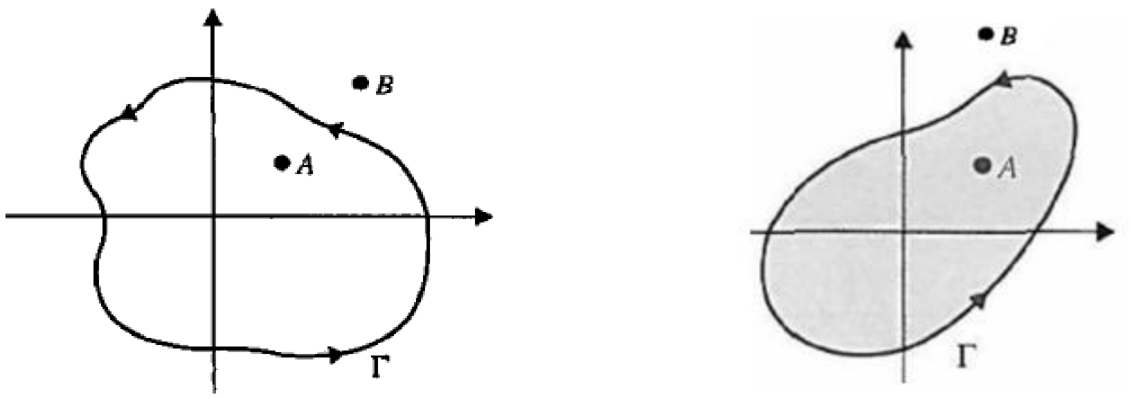


Fig. 5, (a) definition of Encircled (b) definition of Enclosed

<u>Number of Encirclements (N)</u>

When a point is encircled by a closed path Γ, a number *N* can be assigned to the number of times it is encircled. The magnitude of *N* can be determined by drawing an arrow from the point to any arbitrary point s_1 and consider it as starting point, then follow the path in the prescribed direction until it returns to the starting point. The total *net* number of revolutions is *N*, which is positive for CCW encirclement and negative for CW encirclement. For example, point *A* in Fig. 6(a) is *encircled once in CW* direction (N= - 1) and point *B* is *encircled twice in CW* direction (N= - 2). In Fig. 6(b), point *A* is *encircled once in CCW* direction (N=+1), and point *B* is *encircled twice* (N=+2).

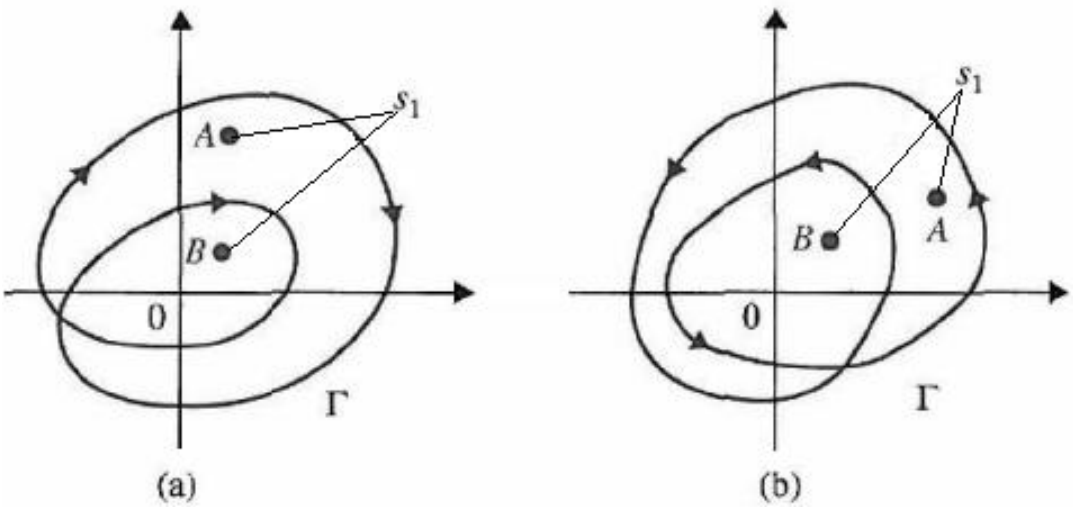


Fig. 6, Definition of number of encirclement (N)

The stability of linear control systems is analysed by constructing the Nyquist path, which is a closed contour in the s plane enclose the entire right-half of s plane. The contour consists of the entire jω axis from ω=−∞ to ∞ and a semi-circular path of infinite radius in the right-half of s plane in the clockwise (CW) or counter clockwise (CCW) direction. Therefore, the Nyquist path encloses the entire right-half s plane and encloses all the zeros and poles of 1+G(s)H(s) that have positive real parts as shown in Fig. 7 (a). The Nyquist path must not pass through poles or zeros of G(s)H(s), if the function G(s)H(s) has poles or zeros at the origin (or at any point on the jω axis), the Nyquist path must be modified by using a semicircle with radius $\varepsilon \rightarrow 0$ as shown in Fig. 7 (b).

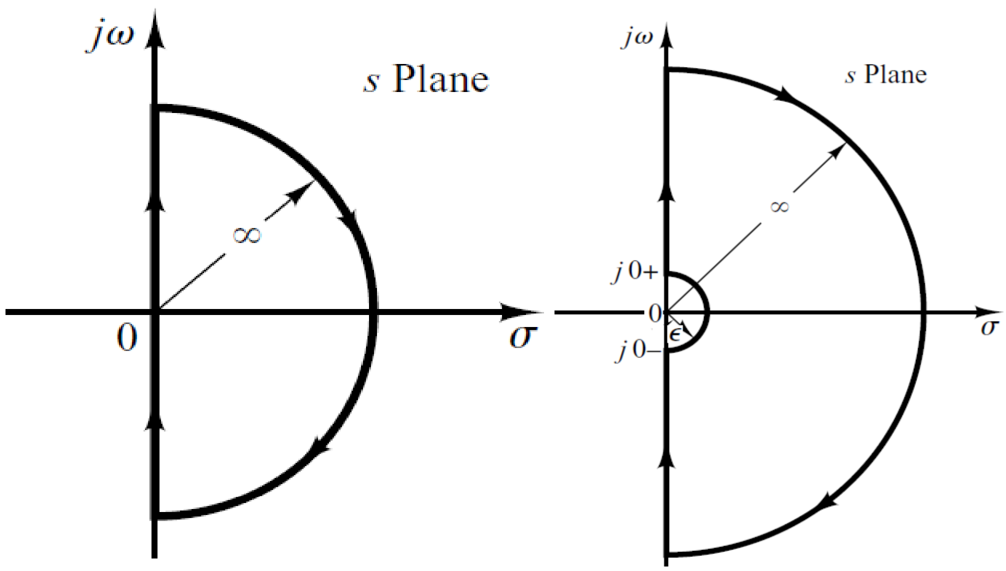


Fig. 7, (a) Nyquist contour (b) Nyquist contour with pole at origin

For the control system shown in Fig. 8,

G(s)H(s) is called open loop T.F.

$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$ is called closed-loop T.F.

1+G(s)H(s) is called characteristic equation

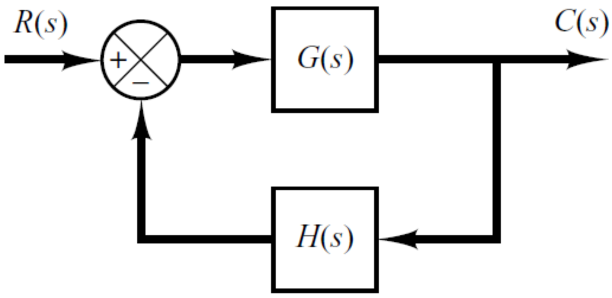


Fig. 8, Closed-loop control system

$$G(s)H(s) = \frac{Z_o}{P_o}$$

$$1 + G(s)H(s) = 1 + \frac{Z_o}{P_o} = \frac{P_o + Z_o}{P_o} = \frac{Z_{-1}}{P_{-1}}$$

It is clear that the open-loop poles = characteristic-equation poles ($P_o = P_{-1}$)

Also, the characteristic equation zeros Z_{-1} = closed-loop T.F. poles

Mapping Theory:

If an open-loop transfer function G(s)H(s) is represented by a ratio of two polynomials as function of s, and P_o be the number of poles and Z_o be the number of zeros of G(s)H(s) that lie inside the closed contour in the s plane. And this contour does not pass through any poles or zeros of G(s)H(s). This closed contour in the s plane is then mapped into the GH plane as a closed curve. The total number of encirclements (N_o) around the origin of GH plane, as a representative point s traces out the entire contour in the clockwise direction, is equal to Z_o-P_o. (Note that by this mapping theory, the numbers of zeros and of poles cannot be found - only their difference.)

$$N_o = Z_o - P_o$$

Where N_o is the number of encirclements around the origin made by the mapped path

Z_o is total number of zeros of GH(s) located at RHS of s-plane

P_o is total number of poles of GH(s) located at RHS of s-plane

As illustrated in Fig. 9, the path in s-plane Γ_S is started at $S_1 \rightarrow S_2 \rightarrow S_3$ …. etc. the mapped path in GH plane Γ_{GH} is also started at $GH(S_1) \rightarrow GH(S_2) \rightarrow GH(S_3)$. The direction of Γ_{GH} can be either CW or CCW, that is, in the same direction or the opposite direction as that of Γ_S depending on the function GH(S).

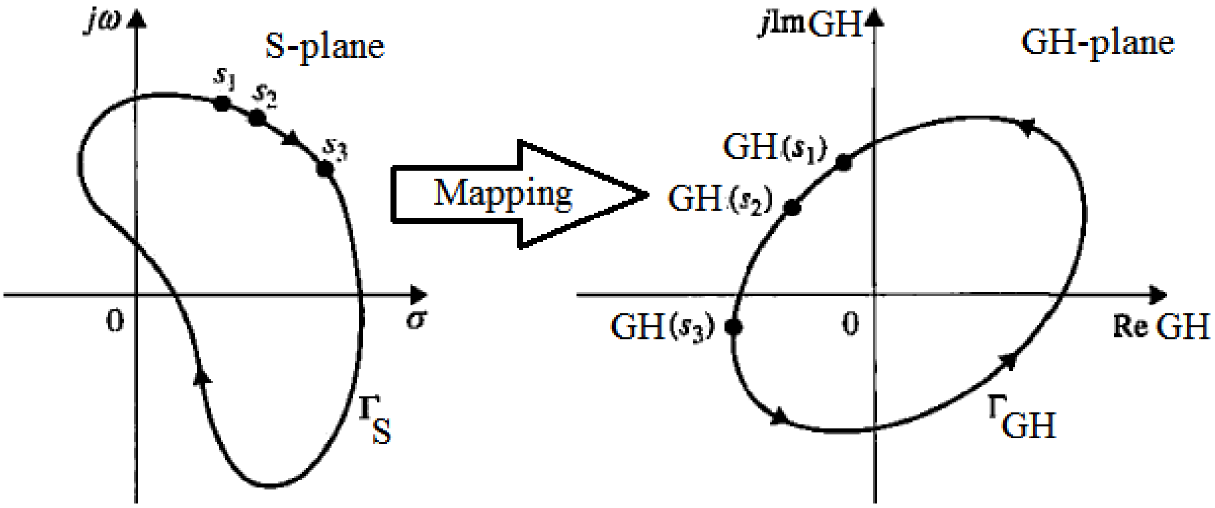


Fig. 9, (a) S-plane contour Γ_S (b) Mapped plot Γ_{GH}

There are 3 possible cases:

1) ($N_o > 0$) $Z_o > P_o$ this means N_o is positive integer. This means the mapped path in GH plane (Γ_{GH}) encircles the origin of GH N_o times in the same direction of Γ_S.
2) ($N_o = 0$) $Z_o = P_o$ this means no encirclement around origin of GH plane
3) ($N_o < 0$) $Z_o < P_o$ this means N_o is negative integer. This means the mapped path in GH plane (Γ_{GH}) encircles the origin of GH N_o times in opposite direction of Γ_S.

The above 3 cases can be summarized in the following table:

Cases of N	N = Z – P	Direction of Γ_S	Number of Encirclements	Direction of Γ_{GH}
1	N > 0	CW CCW	N N	CW CCW
2	N = 0	CW CCW	0 0	No encirclement
3	N < 0	CW CCW	N N	CCW CW

The best way to determine the number of encirclements with respect to origin or any point is to draw a line from that point in any direction to a point out of the mapped path (Γ_{GH}); the number of *net* intersections of this line with the (Γ_{GH}) gives the magnitude of N as shown in Fig. 10.

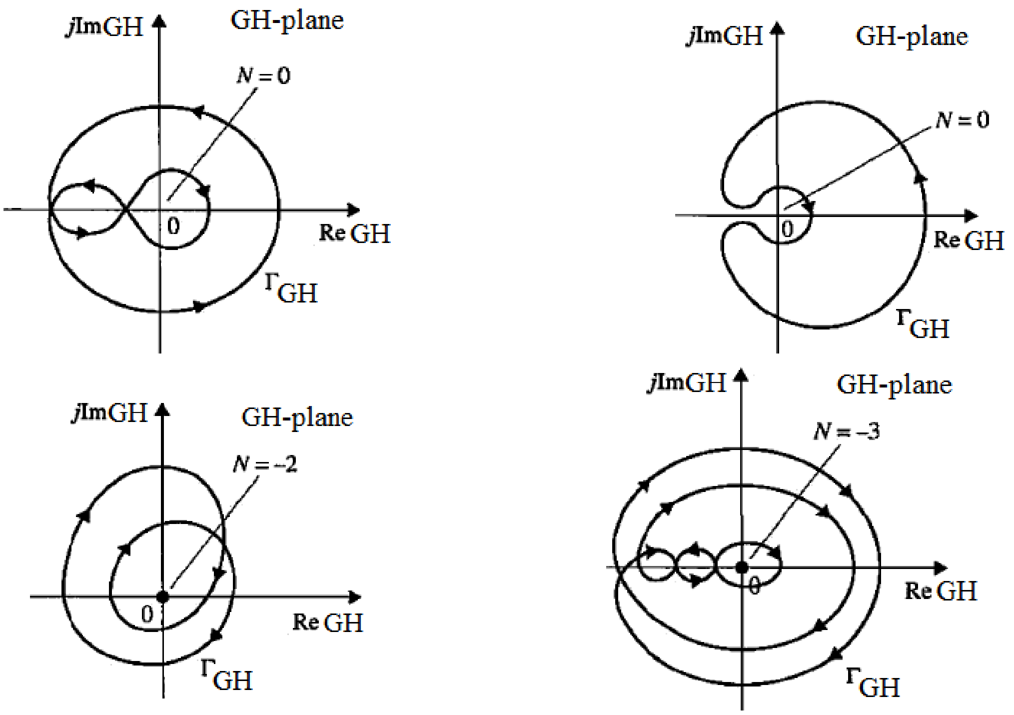


Fig. 10, Determination of number of encirclement

<u>How to map Nyquist contour from S-plane to GH plane?</u>

<u>Example (5):</u>

$$GH(s) = \frac{40}{S(S+2)(S+3)}$$

The Nyquist contour will be as shown in Fig. 11, and we can divide this Nyquist contour into 4 sections.

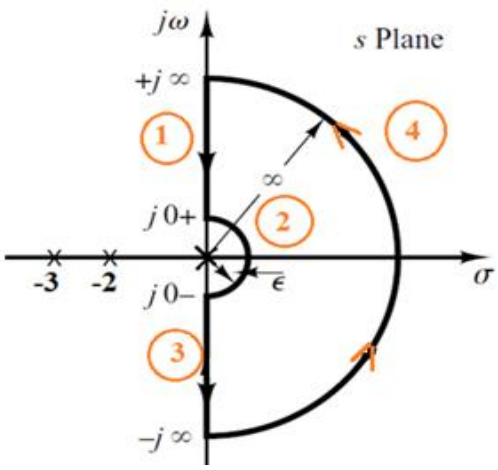


Fig. 11, Nyquist path in S-plane

For section (2),

we replace each S by ε $e^{j\theta}$ (ε→0), (θ take values 90, 45, 0, - 45, - 90)

$$GH(\epsilon e^{j\theta}) = \frac{40}{\epsilon e^{j\theta}(\epsilon e^{j\theta}+2)(\epsilon e^{j\theta}+3)}$$

Since ε $e^{j\theta}$ << 2 so that it can be neglected

Also ε $e^{j\theta}$ << 3 so that it can be neglected, therefore GH function can be rewritten as:

$$GH(\epsilon e^{j\theta}) = \frac{40}{\epsilon e^{j\theta}(2)(3)} = \frac{40}{6\epsilon e^{j\theta}} = \infty e^{-j\theta}$$

For section (4),

we replace each S by R $e^{j\theta}$ (R→0), (θ take values - 90, - 45, 0, 45, 90)

$$GH(Re^{j\theta}) = \frac{40}{Re^{j\theta}(Re^{j\theta}+2)(Re^{j\theta}+3)}$$

Since R $e^{j\theta}$ >> 2 so that the term (2) can be neglected

Also R $e^{j\theta}$ >> 3 so that the term (3) can be neglected, therefore GH function can be rewritten as:

$$GH(Re^{j\theta}) = \frac{40}{Re^{j\theta}(Re^{j\theta})(Re^{j\theta})} = \frac{40}{R^3 e^{j3\theta}} = 0\ e^{-j3\theta}$$

For section (1),

we replace each S by jω

$$G(j\omega)H(j\omega) = \frac{40}{j\omega(2+j\omega)(3+j\omega)} = \frac{40}{-5\omega^2 + j\omega(6-\omega^2)} \times \frac{-5\omega^2 - j\omega(6-\omega^2)}{-5\omega^2 - j\omega(6-\omega^2)}$$

$$= \frac{-200\omega^2}{25\omega^4 + \omega^2(6-\omega^2)^2} - j\frac{40\omega(6-\omega^2)}{25\omega^4 + \omega^2(6-\omega^2)^2}$$

Intersection with real axis

To get the intersection with real, take the imaginary part = 0

$$40\omega(6-\omega^2) = 0 \rightarrow \omega^2 = 6 \ \rightarrow \ \omega_c = 2.45$$

Real distance can be obtained by substituting the value of ω_c in the real part

$$real\ distance = \frac{-200 \times 6}{25 \times 36 + 0} = -1.33333$$

Intersection with imaginary axis

To get the intersection with imaginary, take the real part = 0

$$-200\omega^2 \quad \rightarrow \omega = 0, \qquad (No\ intersection)$$

For section (3),

we replace each S by – jω, and it will give same results as section (1) with opposite sign.

The mapped Nyquist path to GH plane is shown in Fig. 12.

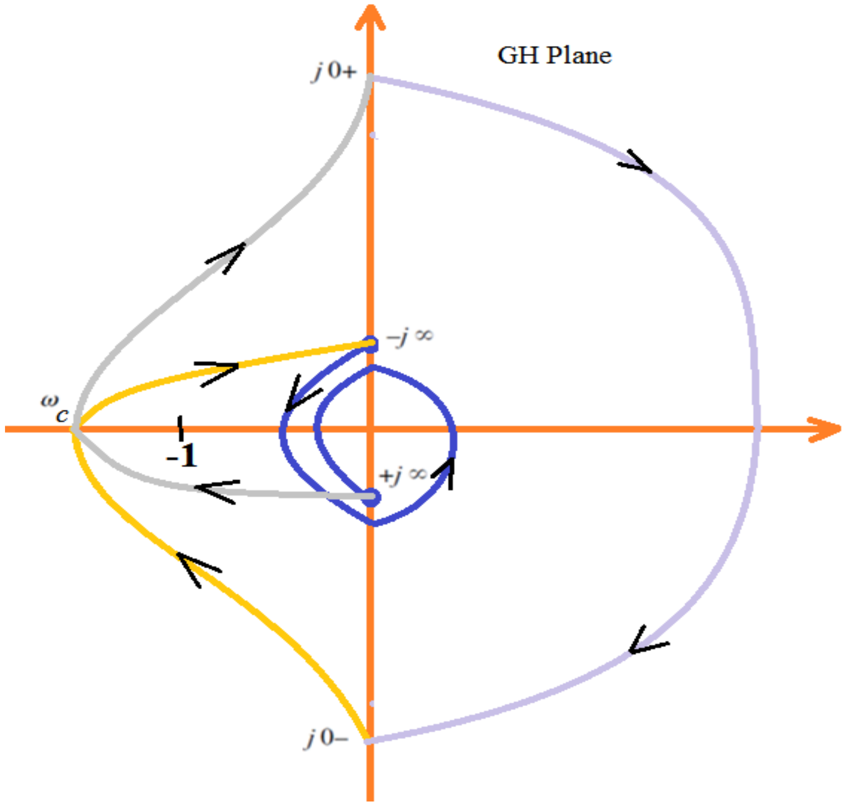


Fig. 12, Nyquist path in GH plane

To discuss the open-loop system stability, it is clear that $Z_o = 0$ and $P_o = 0$. Also, the number of encirclements around the origin $N_o = 0$

Therefore, the equation $N_o = Z_o - P_o$ is satisfied. This means the **open-loop system is stable**.

On the other hand, to discuss the closed-loop system stability, Z_{-1} must be zero

And $P_{-1} = P_o = 0$ (from open-loop system)

Therefore, the number of encirclements N_{-1} around (-1) must satisfy

$N_{-1} = Z_{-1} - P_{-1} = 0 - 0 = 0$

From Fig. 12, we found that $N_{-1} = -2$, therefore **the closed-loop system is unstable**

Example (6):

To stabilize the system given in previous example, a zero (1+Ts) is added, find the value of T based on Nyquist criterion.

$$GH(s) = \frac{40\,(1 + TS)}{S(S+2)(S+3)}$$

The Nyquist contour will be as shown in Fig. 12, and we can divide this Nyquist contour into 4 sections.

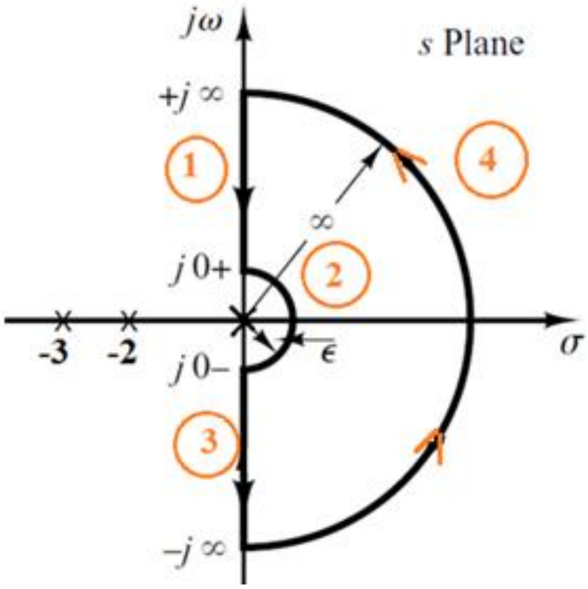


Fig. 12, Nyquist path in S-plane

For section (2),

we replace each S by $\varepsilon\, e^{j\theta}$ ($\varepsilon \to 0$), (θ take values 90, 45, 0, - 45, - 90)

$$GH(\epsilon e^{j\theta}) = \frac{40(\epsilon T e^{j\theta} + 1)}{\epsilon e^{j\theta}(\epsilon e^{j\theta} + 2)(\epsilon e^{j\theta} + 3)}$$

Since $\varepsilon T\, e^{j\theta} << 1$ so that it can be neglected

And $\varepsilon\, e^{j\theta} << 2$ so that it can be neglected

Also $\varepsilon\, e^{j\theta} << 3$ so that it can be neglected, therefore GH function can be rewritten as:

$$GH(\epsilon e^{j\theta}) = \frac{40}{\epsilon e^{j\theta}(2)(3)} = \frac{40}{6\epsilon e^{j\theta}} = \infty e^{-j\theta}$$

For section (4),

we replace each S by $R\, e^{j\theta}$ ($R \to 0$), (θ take values - 90, - 45, 0, 45, 90)

$$GH(Re^{j\theta}) = \frac{40\,(RTe^{j\theta}+1)}{Re^{j\theta}(Re^{j\theta}+2)(Re^{j\theta}+3)}$$

Since RT $e^{j\theta}$ >> 1 so that the term (1) can be neglected

R $e^{j\theta}$ >> 2 so that the term (2) can be neglected

Also R $e^{j\theta}$ >> 3 so that the term (3) can be neglected, therefore GH function can be rewritten as:

$$GH(Re^{j\theta}) = \frac{40\,RTe^{j\theta}}{Re^{j\theta}(Re^{j\theta})(Re^{j\theta})} = \frac{40T}{R^2e^{j2\theta}} = 0\,e^{-j2\theta}$$

For section (1),

we replace each S by jω

$$G(j\omega)H(j\omega) = \frac{40(1+j\omega T)}{j\omega(2+j\omega)(3+j\omega)} = \frac{40(1+j\omega T)}{-5\omega^2+j\omega(6-\omega^2)} \times \frac{-5\omega^2-j\omega(6-\omega^2)}{-5\omega^2-j\omega(6-\omega^2)}$$

$$= \frac{-40\omega^2(5-6T+\omega^2T)}{25\omega^4+\omega^2(6-\omega^2)^2} - j\frac{40\omega(5\omega^2T+6-\omega^2)}{25\omega^4+\omega^2(6-\omega^2)^2}$$

Intersection with real axis

To get the intersection with real, take the imaginary part = 0

$$-40\omega(5\omega^2T+6-\omega^2) = 0 \rightarrow \omega^2 = \frac{6}{1-5T} \rightarrow \omega_c = \sqrt{\frac{6}{1-5T}}$$

Real distance can be obtained by substituting the value of ω_c in the real part

$$real\ distance = \frac{-40[(5-6T)(1-5T)^2+6T(1-5T)]}{36+78(1-5T)+36(1-5T)^2}$$

$$real\ distance = \frac{6000T^3-6200T^2+2000T-200}{900T^2-750T+150}$$

As indicated from the Nyquist plot given in Fig. 13, the closed-loop system to be stable $Z_{-1} = 0$

Also it is known that $P_{-1} = P_o = 0$ (from open loop system)

For the closed-loop system to be stable $N_{-1} = 0 - 0 = 0$

This mean the magnitude of the real distance must be less than or equal 1, or the value of real distance must be greater than or equal –1.

This gives the following inequality:

$$\frac{6000T^3 - 6200T^2 + 2000T - 200}{900T^2 - 750T + 150} \geq -1$$

$$6000T^3 - 5300T^2 + 1250T - 50 \geq 0$$

Solving this inequality gives:

T ≥ 0.05 (accepted) gives real value for ω_c and $\omega_c = \sqrt{8} = 2.83$

T ≥ 0.5 (rejected) doesn't give real value for ω_c

T ≥ 0.3333 (rejected) doesn't give real value for ω_c

Intersection with imaginary axis

To get the intersection with imaginary, take the real part = 0

$$5 - 6T + \omega^2 T \quad \rightarrow \omega = \sqrt{\frac{6T-5}{T}}$$

By substitution of the values of T obtained above, we get imaginary values of ω for all values of T. This means no intersection with the imaginary axis.

For section (3),

we replace each S by – jω, and it will give same results as section (1) with opposite sign. The mapped Nyquist path to GH plane is shown in Fig. 13.

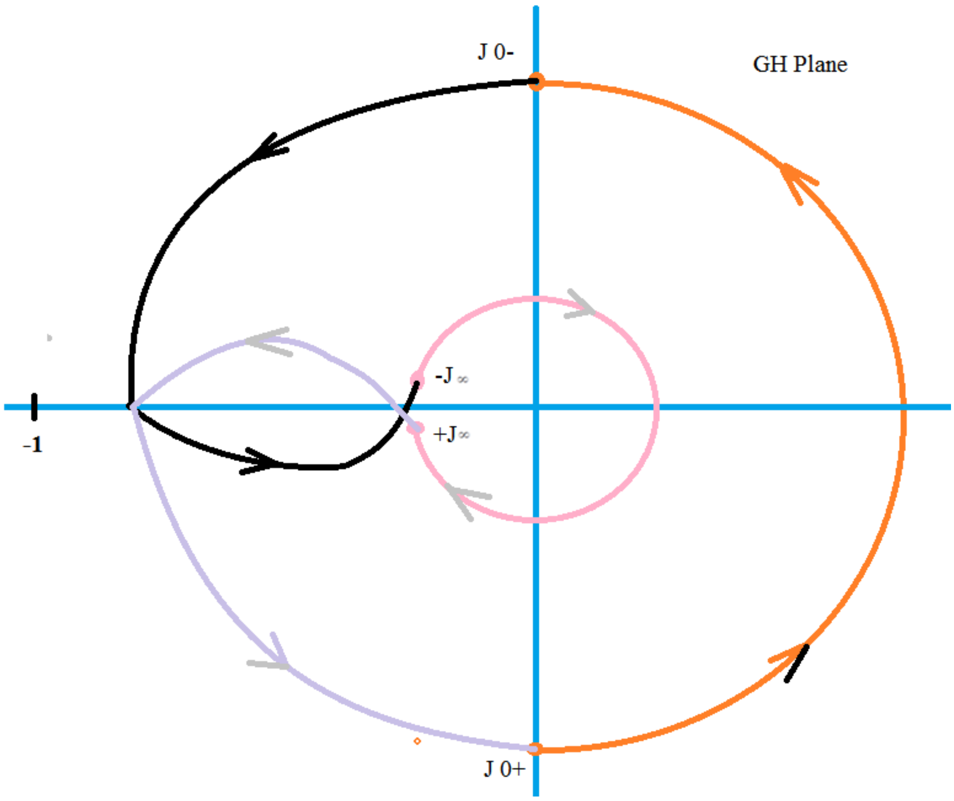


Fig. 13, Nyquist plot in GH plane

As check, using Routh criterion:

The system characteristic equation is:

$$S^3 + 5S^2 + 6S + 40 + 40TS = 0$$

S^3	1	40T+6
S^2	5	40
S^1	40T – 2	
S^0	40	

$40T - 2 \geq 0$
$T \geq 0.05$ # as obtained from Nyquist plot.

Example (7):

Given the characteristic equation of a control system:

$$S^3 + 5S^2 + (K+6)S + 8K = 0$$

Using Nyquist criterion to discuss the system stability.

$$G(S)H(S) = \frac{K(S+8)}{S(S+2)(S+3)}$$

The Nyquist contour will be as shown in Fig. 14, and we can divide this Nyquist contour into 4 sections.

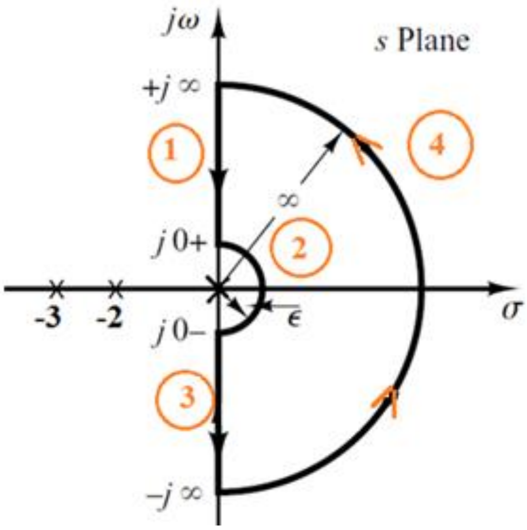


Fig. 14, Nyquist contour in S-plane

For section (2),

we replace each S by $\varepsilon\, e^{j\theta}$ ($\varepsilon \to 0$), (θ take values 90, 45, 0, - 45, - 90)

$$GH(\epsilon e^{j\theta}) = \frac{K(\epsilon e^{j\theta}+8)}{\epsilon e^{j\theta}(\epsilon e^{j\theta}+2)(\epsilon e^{j\theta}+3)}$$

Since ε $e^{j\theta}$ << 8 so that it can be neglected

And ε $e^{j\theta}$ << 2 so that it can be neglected

Also ε $e^{j\theta}$ << 3 so that it can be neglected, therefore GH function can be rewritten as:

$$GH(\epsilon e^{j\theta}) = \frac{8K}{\epsilon e^{j\theta}(2)(3)} = \frac{8K}{6\epsilon e^{j\theta}} = \infty e^{-j\theta}$$

For section (4),

we replace each S by R $e^{j\theta}$ (R→∞), (θ take values - 90, - 45, 0, 45, 90)

$$GH(Re^{j\theta}) = \frac{K\,(Re^{j\theta}+8)}{Re^{j\theta}(Re^{j\theta}+2)(Re^{j\theta}+3)}$$

Since R $e^{j\theta}$ >> 8 so that the term (8) can be neglected

R $e^{j\theta}$ >> 2 so that the term (2) can be neglected

Also R $e^{j\theta}$ >> 3 so that the term (3) can be neglected.

Therefore GH function can be rewritten as:

$$GH(Re^{j\theta}) = \frac{K\,Re^{j\theta}}{Re^{j\theta}(Re^{j\theta})(Re^{j\theta})} = \frac{K}{R^2 e^{j2\theta}} = 0\; e^{-j2\theta}$$

For section (1),

we replace each S by jω

$$G(j\omega)H(j\omega) = \frac{K(8+j\omega)}{j\omega(2+j\omega)(3+j\omega)} = \frac{K(8+j\omega)}{-5\omega^2+j\omega(6-\omega^2)} \times \frac{-5\omega^2-j\omega(6-\omega^2)}{-5\omega^2-j\omega(6-\omega^2)}$$

$$= \frac{-K\omega^2(\omega^2+34)}{\omega^2(25\omega^2+(6-\omega^2)^2)} - j\frac{\omega K(48-3\omega^2)}{\omega^2(25\omega^2+(6-\omega^2)^2)}$$

Intersection with real axis

To get the intersection with real, take the imaginary part = 0

$$48-3\omega^2 \;\rightarrow\; \omega^2 = 16 \;\rightarrow\; \omega_c = 4$$

Real part:

$$\frac{-K(16+34)}{25\times 16+(6-16)^2} = \frac{-50K}{500} = -0.1K$$

Intersection with imaginary axis

To get the intersection with imaginary, take the real part = 0

$$\omega^2 + 34 = 0 \quad \rightarrow \omega^2 = -34, \qquad (No\ intersection)$$

For section (3),

we replace each S by – jω, and it will give same results as section (1) with opposite sign.

The mapped Nyquist path to GH plane is shown in Fig. 15.

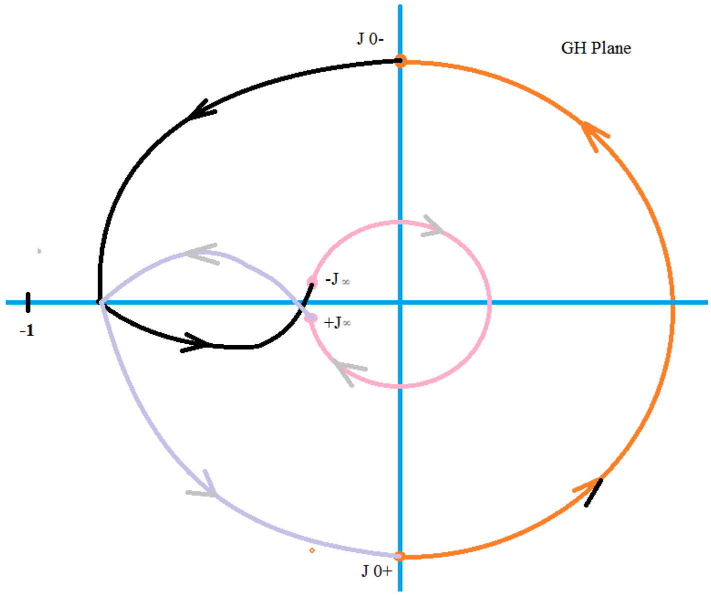


Fig. 15, Nyquist plot in GH plane

As indicated from the Nyquist plot given in Fig. 15, the closed-loop system to be stable $Z_{-1} = 0$, Also it is known that $P_{-1} = P_o = 0$ (from open loop system)

For the closed-loop system to be stable $N_{-1} = 0 – 0 = 0$

This mean the magnitude of the real distance must be less than or equal 1, or the value of real distance must be greater than or equal –1.

This gives the following inequality:

$$0.1K \leq 1 \quad \rightarrow \quad K \leq 10$$

Check using Routh:

the characteristic equation of a control system:

$$S^3 + 5S^2 + (K+6)S + 8K = 0$$

S^3	1	6+K
S^2	5	8K
S^1	(30-3K)/5	
S^0	8K	

$30 – 3K \geq 0 \quad \rightarrow \quad K \leq 10$

Auxiliary equation:

$5S^2 + 80 = 0 \quad \rightarrow \quad S = -16 \quad \rightarrow \quad S = \pm J\,4$ which is considered as ω_c #

Phase Margin and Gain Margin:

There are two principal measures of system stability determined via frequency methods: Gain Margin, and Phase Margin, whereby the degree of stability or instability may be quantified. This is done by measuring how close the Nyquist plot to (-1).

Why (-1), because the characteristic equation $1+GH = 0 \rightarrow GH = -1$

This means $|GH| = 1$ and $\angle GH = -180$

Nyquist plot of *GH* intersects the negative real axis at a point (ω_c) called phase crossover frequency at which the angle of $GH = -180$

Gain Margin (GM) is one of the most frequently used criteria for measuring relative stability of control systems. In the frequency domain, gain margin is used to indicate the closeness of the intersection of the negative real axis made by the Nyquist plot to the −1.0 point.

G.M. is defined as the amount of gain that can be added to the open-loop T.F (GH) before the closed-loop system becomes unstable.

How to calculate G.M. from polar plot:

Step #1: replace $s \rightarrow j\omega$ in the open-loop T.F. to obtain $GH(j\omega)$

Step #2: take the imaginary part and equate by zero to obtain the phase crossover frequency ω_c.

Step #3: calculate the real distance by substituting ω_c in the real part of $GH(j\omega)$.

Step #4: Calculate the G.M. from the following formula:

$$G.M. = \frac{1}{|real\ distance|_{\omega=\omega_c}}$$

Phase Margin Φ_{PM}: gain margin alone is inadequate to indicate relative stability when system parameters other than the loop gain are subject to variation.

The definition of phase margin (Φ_{PM}) is the angle in degrees through which the GH plot must be rotated about the origin so that the gain crossover passes through the -1 point.

Gain Crossover: The gain crossover is a point on the GH plot at which the magnitude of GH(jω) is equal to 1. The frequency at this point is called **Gain Crossover Frequency** (ω_1).

How to calculate Phase Margin Φ_{PM}:

Step #1: replace s → jω in the open-loop T.F. to obtain GH(jω)

Step #2: take the absolute value of the real part of GH(jω) and equate by one to obtain the gain crossover frequency ω_1.

Step #3: Calculate the angle GH(jω) that is $\angle GH(j\omega)|_{\omega = \omega 1}$

Step #4: Calculate the Phase Margin $\Phi_{PM} = 180 + \angle GH(j\omega)|_{\omega = \omega 1}$

The graphical representation of calculation of both gain and phase margins in case of stable system is shown in Fig. 16. Where the GM > 1 and P.M is +ve

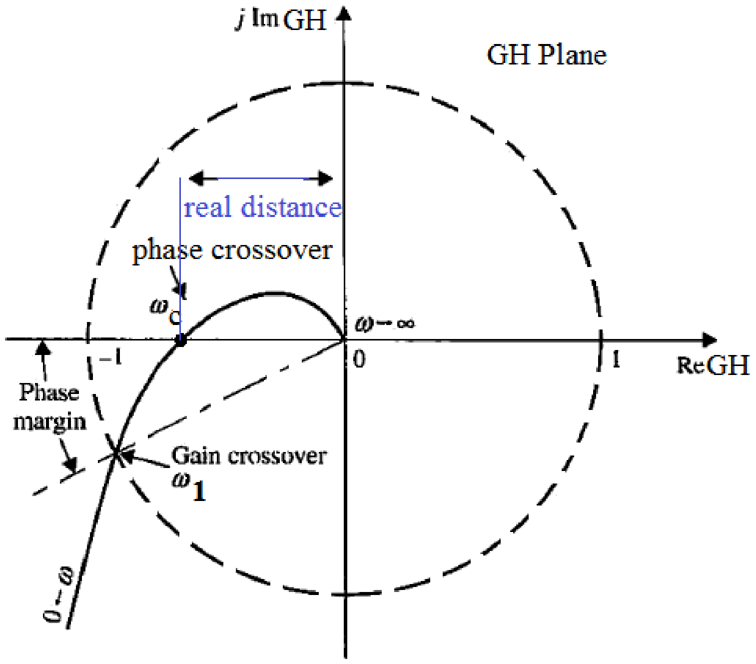


Fig. 16, Stable system

The graphical representation of calculation of both gain and phase margins in case of stable system is shown in Fig. 17. Where the GM < 1 and P.M is negative

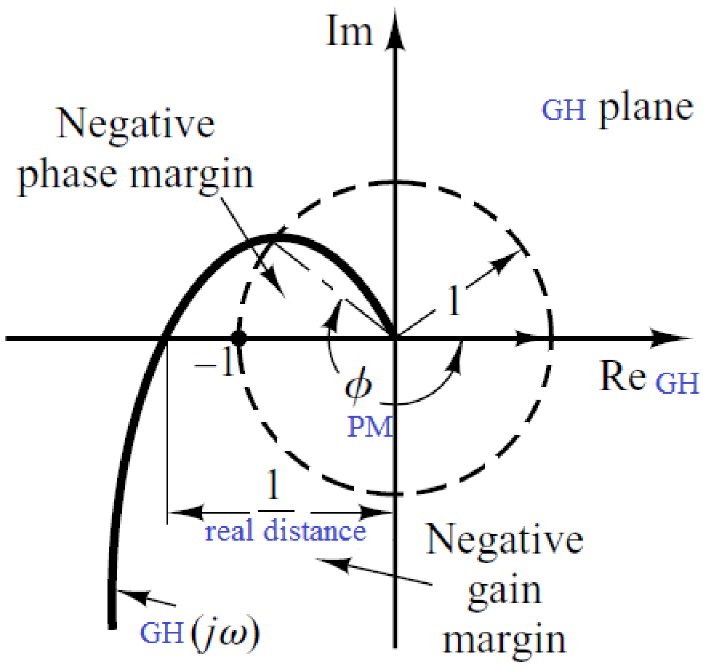


Fig. 17 Unstable system

In case of GM =1 and Phase margin = 0, the system is marginally stable as shown in Fig. 18.

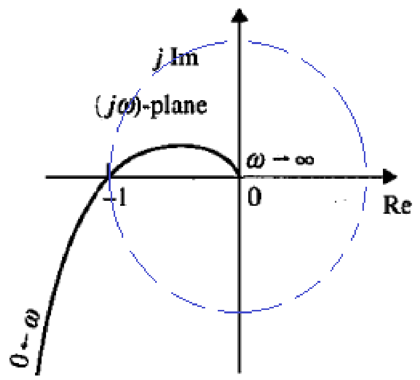


Fig. 18 Marginally stable system

Advantages of the Nyquist Plot

The stability analysis of a closed-loop system can be easily investigated by examining the Nyquist plot of the loop transfer function with reference to -1 point.

Sheet 11 (Nyquist Plots)

(1) Draw the Nyquist plot for the unity-feedback control system with the open-loop transfer function:

$$GH = \frac{K(1-S)}{S+1}$$

Using the Nyquist stability criterion, determine the stability of the closed-loop system.

(2) Draw the Nyquist plot for the unity-feedback control system with the open-loop transfer function:

$$GH = \frac{K(S+3)}{S(S-1)}$$

Using the Nyquist stability criterion, determine the stability of the closed-loop system.

(3) Draw the Nyquist plot for the unity-feedback control system with the open-loop transfer function:

$$GH = \frac{K}{S(S+5)}$$

Using the Nyquist stability criterion, determine the stability of the closed-loop system.

(4) Draw the Nyquist plot for the unity-feedback control system with the open-loop transfer function:

$$GH = \frac{K(S+1)}{S^2(S+9)}$$

Using the Nyquist stability criterion, determine the stability of the closed-loop system.

(5) Draw the Nyquist plot for the unity-feedback control system with the open-loop transfer function:

$$GH = \frac{2500}{S(S+5)(S+50)}$$

Using the Nyquist stability criterion, determine the stability of the closed-loop system, then calculate:

a) Phase crossover frequency [15.88 rad/s]
b) Gain crossover frequency [6.22 rad/s]
c) Gain margin [5.495]
d) Phase margin [31.72°]

(6) For the unity-feedback control system with the open-loop transfer function:

$$GH = \frac{50}{S(1+0.1S)(1+0.2S)}$$

Using the Nyquist stability criterion, determine the stability of the closed-loop system, then calculate:

a) Phase crossover frequency [7.071 rad/s]
b) Gain crossover frequency [rad/s]
c) Gain margin [0.3]
d) Phase margin [ ]

(7) Draw the Nyquist plot for the unity-feedback control system with the open-loop transfer function:

$$GH = \frac{100}{S(S+1)(S^2+2S+2)}$$

Using the Nyquist stability criterion, determine the stability of the closed-loop system

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